

Grade Level/Course: Algebra 1

Lesson/Unit Plan Name: Average Rate of Change

Rationale/Lesson Abstract: Students will be able to find the average rate of change of a linear function over an interval algebraically, graphically and using a table, and interpret their results. Additional examples are provided involving quadratic and exponential functions.

Timeframe: 60 minutes for Average Rate of Change for Linear Functions (pages 2 – 6)
45 minutes for Average Rate of Change for Quadratic Functions (pages 7 – 9)
30 minutes for Average Rate of Change for Exponential Functions (includes Linear and Quadratic) (pages 10 – 12)

Common Core Standard(s): F-IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

Note: Warm Up for lesson on pages 2 – 6 is located on page 13

Lesson:

The average rate of change of a function over a specified interval gives us a sense of whether the function values are mostly increasing or mostly decreasing on the interval. The formula will look familiar to you.

The Average Rate of Change of a function $f(x)$ from $x = a$ to $x = b$ is

$$\frac{f(b) - f(a)}{b - a} \quad \frac{\text{change in function values}}{\text{change in } x}$$

Does the formula look familiar?

Let's investigate with a linear function.

Example 1)

Consider the linear function $f(x) = 2x - 1$ whose graph is shown.

Find the average rate of change of $f(x) = 2x - 1$ from $x = 0$ to $x = 1$.

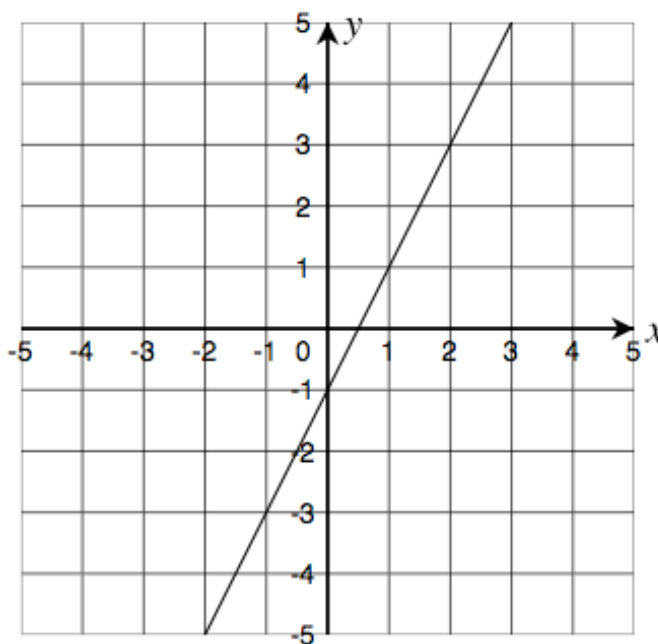
$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = 0, b = 1$$

$$\frac{f(1) - f(0)}{1 - 0}$$

$$= \frac{1 - (-1)}{1}$$

$$= \frac{2}{1}$$

$$= 2$$



You-Try!

Find the average rate of change of $f(x) = 2x - 1$ from $x = -1$ to $x = 2$.

Answer to you-try:

$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = -1, b = 2$$

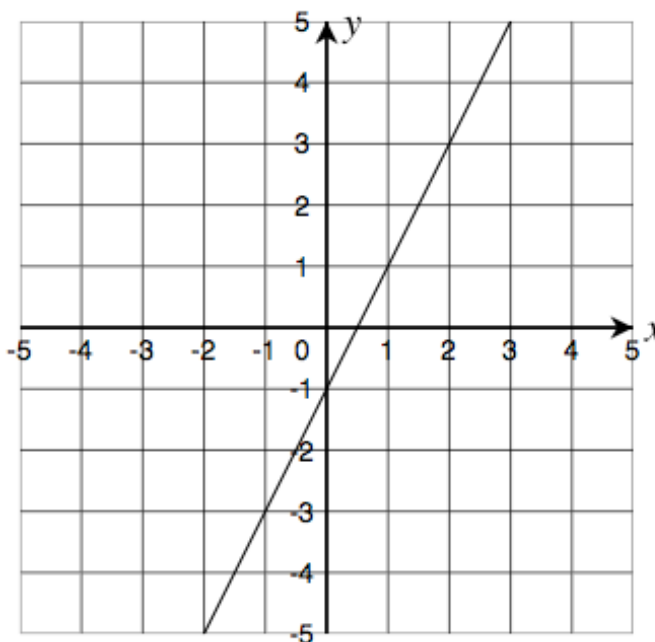
$$\frac{f(2) - f(-1)}{2 - (-1)}$$

$$= \frac{3 - (-3)}{2 + 1}$$

$$= \frac{3 + 3}{3}$$

$$= \frac{6}{3}$$

$$= 2$$



Since the function is linear, the average rate of change is constant. In this case, the average rate of change is constantly 2. It is the slope of the line. Since the average rate of change is positive and constant, this tells us that the function values are increasing over the given intervals.

Remember to read graphs from left to right.

$f(x) = 2x - 1$ is a linear function and therefore has a constant average rate of change. The graph of the line is constantly increasing from left to right which means the function values are getting larger from left to right.

When a function is linear, we often refer to the average rate of change as simply the rate of change.

Later, we will look at the average rate of change of functions that are not linear.

Example 2) (parts a, b and c – parts b and c are partner work)

Using the table, find the average rate of change over the following intervals.

| x | $f(x)$ |
|-----|--------|
| 0 | 5 |
| 1 | 1 |
| 2 | -3 |
| 3 | -7 |
| 4 | -11 |

a) from $x = 0$ to $x = 2$

$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = 0, b = 2$$

$$\begin{aligned} & \frac{f(2) - f(0)}{2 - (0)} \\ &= \frac{-3 - 5}{2 - 0} \\ &= \frac{-8}{2} \\ &= -4 \end{aligned}$$

Partner Work:

Now, with your elbow partner, one of you find the average rate of change for part b and the other find the average rate of change for part c. When finished, write down what the average rate of change tells you about the function in the table.

b) from $x = 1$ to $x = 4$

c) from $x = 3$ to $x = 4$

Answers for partner work:

b) from $x = 1$ to $x = 4$

$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = 1, b = 4$$

$$\begin{aligned} & \frac{f(4) - f(1)}{4 - 1} \\ &= \frac{-11 - 1}{3} \\ &= \frac{-12}{3} \\ &= -4 \end{aligned}$$

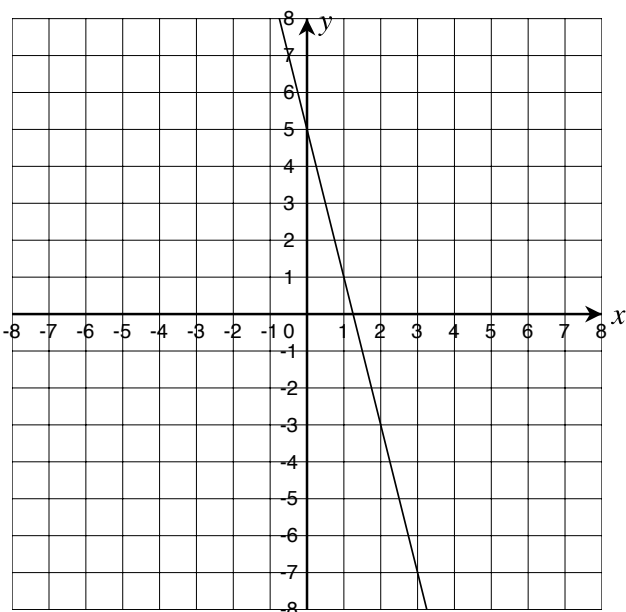
c) from $x = 3$ to $x = 4$

$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = 3, b = 4$$

$$\begin{aligned} & \frac{f(4) - f(3)}{4 - 3} \\ &= \frac{-11 - (-7)}{1} \\ &= \frac{-11 + 7}{1} \\ &= \frac{-4}{1} \\ &= -4 \end{aligned}$$

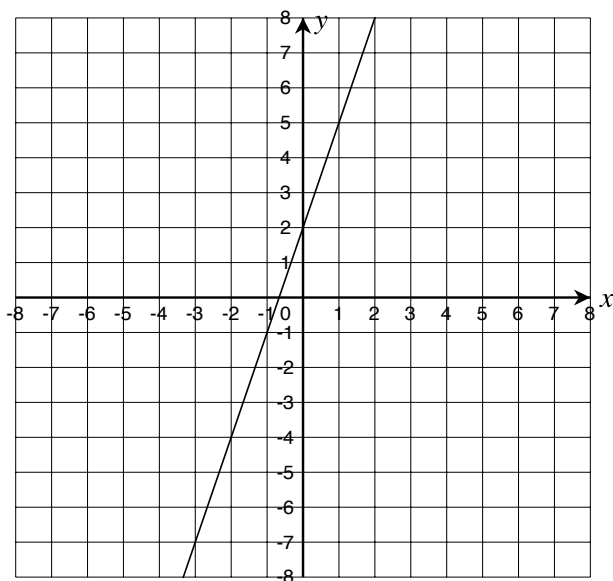
The average rate of change is -4 for parts a, b and c which makes me think that the function is linear since the average rate of change is constantly -4 . Since the average rate of change is constant and negative, I know that the function is decreasing and the function values are getting smaller from left to right.

Some students may draw the graph to help them find the average rate of change.



Exit Ticket:

The graph of a function, $f(x)$, is shown below. Use the graph to decide if the following statements about the average rate of change of $f(x)$ are true or false.



A) $\frac{f(1) - f(-2)}{1 - (-2)}$

True False

B) $\frac{f(1) - f(0)}{1 - 0}$

True False

C) $\frac{f(1) - f(-2)}{1 - 2}$

True False

D) $\frac{f(2) - f(-1)}{2 + 1}$

True False

E) The average rate of change is positive.

True False

F) The average rate of change is negative.

True False

G) The average rate of change is constant.

True False

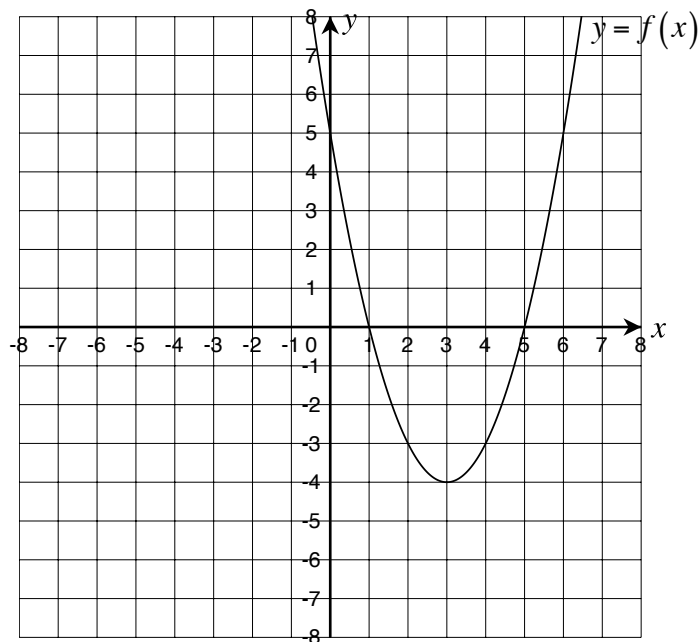
Lesson continued:

Answers to Exit Ticket: T T F T T F T

Average Rate of Change including Quadratic and Exponential Functions

Example 1 (parts a, b and c – part b is a you-try)

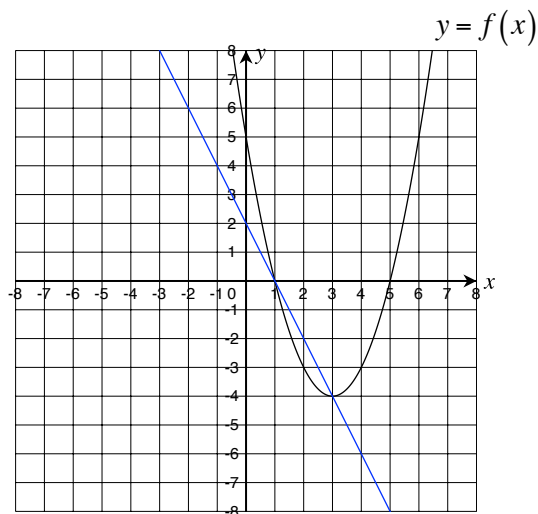
Consider the quadratic function whose graph is shown.



a) Find the average rate of change of $f(x)$ from $x = 1$ to $x = 3$.

$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = 1, b = 3$$

$$\begin{aligned} & \frac{f(3) - f(1)}{3 - 1} \\ &= \frac{-4 - 0}{2} \\ &= \frac{-4}{2} \\ &= -2 \end{aligned}$$



The average rate of change finds the slope of the line through the points $(1, 0)$ and $(3, -4)$ as shown. Since the average rate of change is negative, we know that the function values for the quadratic function, $y = f(x)$ are mostly decreasing on the interval from $x = 1$ to $x = 3$.

You-Try!

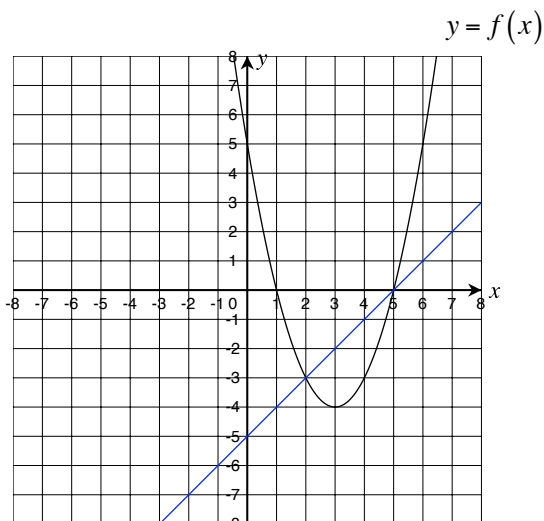
Using the graph from Example 1 complete part b:

- b) Find the average rate of change of $f(x)$ from $x = 2$ to $x = 5$. Does your answer tell you that the function $y = f(x)$ is mostly increasing or mostly decreasing on the interval from $x = 2$ to $x = 5$?

Answer to you-try:

$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = 2, b = 5$$

$$\begin{aligned} & \frac{f(5) - f(2)}{5 - 2} \\ &= \frac{0 - (-3)}{3} \\ &= \frac{3}{3} \\ &= 1 \end{aligned}$$



The average rate of change is positive so the function is mostly increasing on the interval from $x = 2$ to $x = 5$.

Now, show the graph to the students and then discuss how the graph of the parabola decreases from $x = 2$ to $x = 3$ and then increases from $x = 3$ to $x = 5$ which shows that overall, the graph is more increasing than it is decreasing over the entire interval from $x = 2$ to $x = 5$.

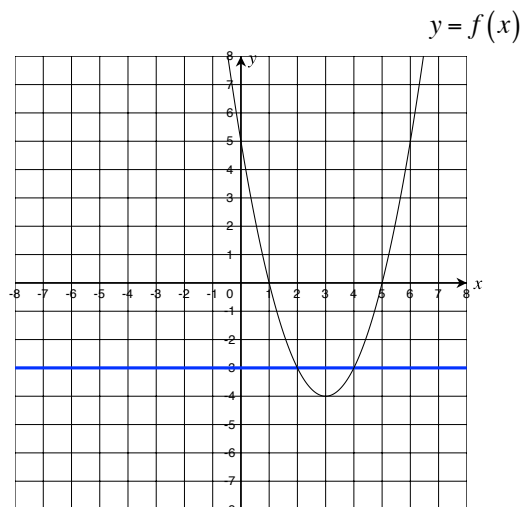
Let's do part c together:

- c) Find the average rate of change of $f(x)$ from $x = 2$ to $x = 4$.

$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = 2, b = 4$$

$$\begin{aligned} & \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{-3 - (-3)}{2} \\ &= \frac{-3 + 3}{2} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

Since the average rate of change is equal to zero, the quadratic function increases and decreases over the interval from $x = 2$ to $x = 4$ but does so equally as you can see by looking at the graph.



Example 2)

Given the function $f(x) = -x^2 + 4x - 1$, find the average rate of change from $x = 1$ to $x = 5$ and interpret what the average rate of change tells you about the function.

$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = 1, b = 5$$

$$\begin{aligned} & \frac{f(5) - f(1)}{5 - 1} \\ &= \frac{-6 - 2}{4} \\ &= \frac{-8}{4} \\ &= -2 \end{aligned}$$

Side work for finding function values:

| | |
|----------------------------|----------------------------|
| $f(x) = -x^2 + 4x - 1$ | $f(x) = -x^2 + 4x - 1$ |
| $f(5) = -(5)^2 + 4(5) - 1$ | $f(1) = -(1)^2 + 4(1) - 1$ |
| $= -25 + 20 - 1$ | $= -1 + 4 - 1$ |
| $= -5 - 1$ | $= 3 - 1$ |
| $= -6$ | $= 2$ |

Since the average rate of change is negative, this tells us that the function is mostly decreasing on the interval from $x = 1$ to $x = 5$.

You-Try!

Given the function $f(x) = -x^2 - 6x + 2$, find the average rate of change from $x = -5$ to $x = -2$ and interpret what the average rate of change tells you about the function.

Answer to you-try:

$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = -5, b = -2$$

$$\begin{aligned} & \frac{f(-2) - f(-5)}{-2 - (-5)} \\ &= \frac{10 - 7}{-2 + 5} \\ &= \frac{3}{3} \\ &= 1 \end{aligned}$$

Side work for finding function values:

| | |
|-------------------------------|-------------------------------|
| $f(x) = -x^2 - 6x + 2$ | $f(x) = -x^2 - 6x + 2$ |
| $f(-5) = -(-5)^2 - 6(-5) + 2$ | $f(-2) = -(-2)^2 - 6(-2) + 2$ |
| $= -25 + 30 + 2$ | $= -4 + 12 + 2$ |
| $= 5 + 2$ | $= 8 + 2$ |
| $= 7$ | $= 10$ |

Since the average rate of change is positive, this tells us that the function is mostly increasing on the interval from $x = -5$ to $x = -2$.

Partner Work:

With your elbow partner, find the average rate of change for the following functions over the given interval. Then, answer the conclusion question.

a) $f(x) = 3x + 5$ from $x = 2$ to $x = 3$

b) $g(x) = x^2 + 4$ from $x = 2$ to $x = 3$

c) $h(x) = 3^x$ from $x = 2$ to $x = 3$

d) Write a conclusion regarding how you can use the average rate of change to compare the three functions on the given interval.

Answers for partner work:

a) $f(x) = 3x + 5$ from $x = 2$ to $x = 3$

b) $g(x) = x^2 + 4$ from $x = 2$ to $x = 3$

$$\frac{f(b) - f(a)}{b - a} \quad \text{note: } a = 2, b = 3$$

$$\frac{g(b) - g(a)}{b - a} \quad \text{note: } a = 2, b = 3$$

$$\begin{aligned} & \frac{f(3) - f(2)}{3 - 2} \\ &= \frac{14 - 11}{1} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

$$\begin{aligned} & \frac{g(3) - g(2)}{3 - 2} \\ &= \frac{13 - 8}{1} \\ &= \frac{5}{1} \\ &= 5 \end{aligned}$$

c) $h(x) = 3^x$ from $x = 2$ to $x = 3$

$$\frac{h(b) - h(a)}{b - a} \quad \text{note: } a = 2, b = 3$$

$$\begin{aligned} & \frac{h(3) - h(2)}{3 - 2} \\ &= \frac{27 - 9}{1} \\ &= \frac{18}{1} \\ &= 18 \end{aligned}$$

d) Since the average rate of change for all functions is positive, this shows that the function values are mostly increasing on the given interval. The exponential function is increasing faster than the quadratic function which is increasing faster than the linear function.

You-Try!

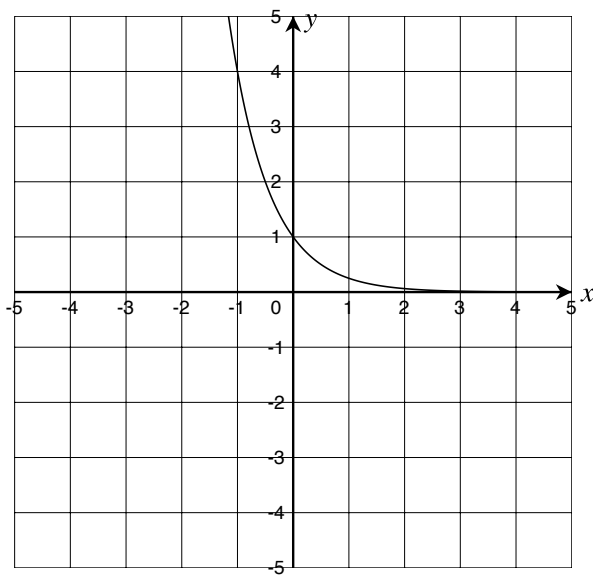
Find the average rate of change for the following functions over the given interval. What does the average rate of change tell you about the function on the interval?

a)

| | | | | | |
|--------|---|---|----|----|----|
| x | 0 | 1 | 2 | 3 | 4 |
| $f(x)$ | 3 | 6 | 12 | 24 | 48 |

From $x = 0$ to $x = 3$

b)



From $x = -1$ to $x = 1$

Answers to you-try:

a)

$$\frac{f(b)-f(a)}{b-a} \quad \text{note: } a=0, b=3$$

$$\begin{aligned} & \frac{f(3)-f(0)}{3-0} \\ &= \frac{24-3}{3} \\ &= \frac{21}{3} \\ &= 7 \end{aligned}$$

Since the average rate of change is positive, the function is mostly increasing on the given interval.

b)

$$\frac{f(b)-f(a)}{b-a} \quad \text{note: } a=-1, b=1$$

$$\begin{aligned} & \frac{f(1)-f(-1)}{1-(-1)} \\ &= \frac{\frac{1}{4}-4}{1+1} \\ &= \frac{\frac{1}{4}-4}{2} \cdot \frac{4}{4} \\ &= \frac{1-16}{8} \\ &= -\frac{15}{8} \end{aligned}$$

Since the average rate of change is negative, the function is mostly decreasing on the given interval.

Warm-Up

Review: CA Alg. 1 CCSS F-IF.2

Select all of the following that are true about the given function.

$$f(x) = -2x - 5$$

- A) $f(0) = -5$
- B) $f(-1) = -7$
- C) $f(-2) = -1$
- D) $f(1) = -7$

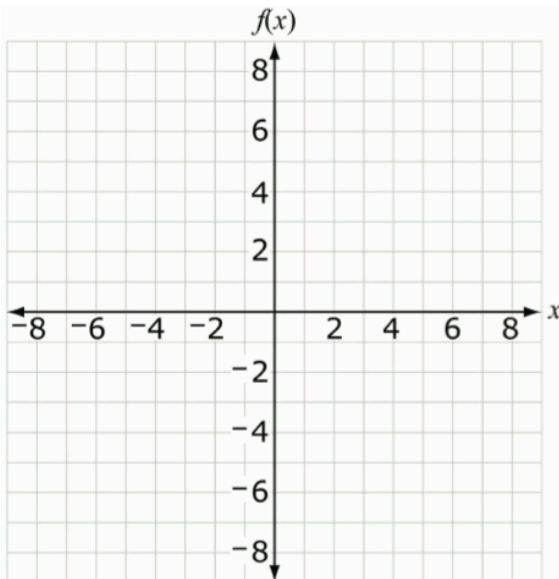
Review: CA Alg. 1 CCSS A-REI.3

Solve the equation using three different methods. Include a visual model.

$$5x + 2 = 3x + 14$$

Current: CA Alg. 1 CCSS F-IF.7a

Draw the graph of the function $f(x) = -\frac{3}{2}x - 3$ on the coordinate plane below.



Current: CA Alg. 1 CCSS F-IF.8

Write the function $y - 3 = \frac{2}{3}(x - 4)$ in the equivalent form most appropriate for identifying the slope and y-intercept of the function.

Answers to Warm Up:

Quadrant I $5x + 2 = 3x + 14$

| <u>Inverse Operations</u> | <u>Decomposition</u> | <u>Bar Model</u> | | | | | | | | | | | | | | | | | | | | |
|---|--|--|----------|--|--|-----------|--|--|------|------|-----|------|------|-----|------|-----|-----|-----|------|-----|-----|-----|
| $5x + 2 = 3x + 14$ $5x - 3x + 2 = 3x - 3x + 14$ $2x + 2 = 14$ $2x + 2 - 2 = 14 - 2$ $2x = 12$ $\frac{2x}{2} = \frac{12}{2}$ $x = 6$ | $5x + 2 = 3x + 14$ $3x + 2x + 2 = 3x + 14$ $2x + 2 = 14$ $2x + 2 = 2 + 12$ $2x = 12$ $x + x = 6 + 6$ $x = 6$ | <table border="1"><tr><td colspan="3">$5x + 2$</td></tr><tr><td colspan="3">$3x + 14$</td></tr></table> <table border="1"><tr><td>$3x$</td><td>$2x$</td><td>2</td></tr><tr><td>$3x$</td><td>12</td><td>2</td></tr></table> <table border="1"><tr><td>$3x$</td><td>x</td><td>x</td><td>2</td></tr><tr><td>$3x$</td><td>6</td><td>6</td><td>2</td></tr></table> <p>$\therefore x = 6$</p> | $5x + 2$ | | | $3x + 14$ | | | $3x$ | $2x$ | 2 | $3x$ | 12 | 2 | $3x$ | x | x | 2 | $3x$ | 6 | 6 | 2 |
| $5x + 2$ | | | | | | | | | | | | | | | | | | | | | | |
| $3x + 14$ | | | | | | | | | | | | | | | | | | | | | | |
| $3x$ | $2x$ | 2 | | | | | | | | | | | | | | | | | | | | |
| $3x$ | 12 | 2 | | | | | | | | | | | | | | | | | | | | |
| $3x$ | x | x | 2 | | | | | | | | | | | | | | | | | | | |
| $3x$ | 6 | 6 | 2 | | | | | | | | | | | | | | | | | | | |

Quadrant II

$$f(x) = -2x - 5$$

A)

$$f(0) = -2(0) - 5$$
$$= -5$$

B)

$$f(-1) = -2(-1) - 5$$
$$= 2 - 5$$
$$= -3$$

C)

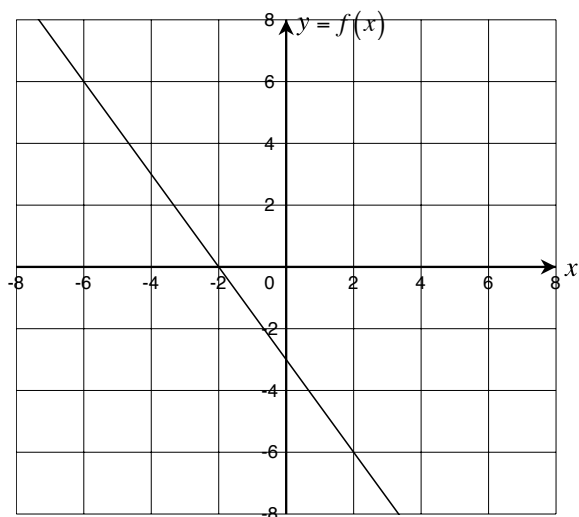
$$f(-2) = -2(-2) - 5$$
$$= 4 - 5$$
$$= -1$$

D)

$$f(1) = -2(1) - 5$$
$$= -2 - 5$$
$$= -7$$

Correct choices: A, C, D

Quadrant III



Quadrant VI

$$y - 3 = \frac{2}{3}(x - 4)$$

$$y - 3 = \frac{2}{3}(x) + \frac{2}{3}(-4)$$

$$y - 3 = \frac{2}{3}x - \frac{8}{3}$$

$$y - 3 + 3 = \frac{2}{3}x - \frac{8}{3} + 3$$

$$y = \frac{2}{3}x - \frac{8}{3} + \frac{9}{3}$$

$$y = \frac{2}{3}x + \frac{1}{3}$$